from which we can define the following double summation:

$$S^l_{ijk,p\bar{n}n} = \sum_{q=-\infty}^{\infty} \sum_{\bar{q}=-\infty}^{\infty} \bar{J}^l_{jqnp} \bar{J}^l_{k\bar{q}\bar{n}p} I^l_{i\bar{q}\bar{n}qn} \tag{A8}$$

where $i = 1, 2, 3$, $j = 0, 1, 2, 3$, and $k = 0, 1, 2, 3$.

With the above definitions, the elements of the 16 submatrices of $[R_l]$, as given by (20), are then as follows, where, due to the orthogonality shown in $I^l_{1,p\bar{p}}$ and $I^l_{2,p\bar{p}}$, by (A1) and (A2), respectively, we can for a given p write

$$C^{(ee)}_{pl,\bar{n}n} = \frac{k_p^2 \bar{R}_l w}{2j\omega\varepsilon_0} S^l_{101,p\bar{n}n}(1+\delta_{p0}) \tag{A9}$$

$$C^{(eh)}_{pl,\bar{n}n} = -\frac{\pi k_p^2 p}{2k_0^2} \left\{ S^l_{220,p\bar{n}n}(1+\delta_{p0}) + S^l_{202,p\bar{n}n}(1+\delta_{p0}) \right\} \tag{A10}$$

$$C^{(hh)}_{pl,\bar{n}n} = \frac{k_p^3 \bar{R}_l w}{2j\omega\mu_0} S^l_{110,p\bar{n}n}(1+\delta_{p0}) \tag{A11}$$

$$S^{(ee)}_{pl,\bar{n}n} = \frac{k_p^3 \bar{R}_l w}{2j\omega\varepsilon_0} S^l_{301,p\bar{n}n}(1+\delta_{p0}) \tag{A12}$$

$$S^{(eh)}_{pl,\bar{n}n} = -\frac{\pi k_p^2 p}{2k_0^2} \left\{ S^l_{221,p\bar{n}n}(1+\delta_{p0}) + S^l_{202,p\bar{n}n}(1+\delta_{p0}) \right\} \tag{A13}$$

$$S^{(hh)}_{pl,\bar{n}n} = \frac{k_p^3 \bar{R}_l w}{2j\omega\mu_0} S^l_{310,p\bar{n}n}(1+\delta_{p0}) \tag{A14}$$

$$K^{(ee)}_{pl,\bar{n}n} = \frac{k_p^2 \bar{R}_l w}{2j\omega\varepsilon_0} S^l_{201,p\bar{n}n}(1+\delta_{p0}) \tag{A15}$$

$$K^{(eh)}_{pl,\bar{n}n} = -\frac{\pi k_p^2 p}{2k_0^2} \left\{ S^l_{320,p\bar{n}n}(1-\delta_{p0}) - S^l_{102,p\bar{n}n}(1+\delta_{p0}) \right\} \tag{A16}$$

$$K^{(hh)}_{pl,\bar{n}n} = \frac{k_p^3 \bar{R}_l w}{2j\omega\mu_0} S^l_{210,p\bar{n}n}(1-\delta_{p0}) \tag{A17}$$

$$Q^{(ee)}_{pl,\bar{n}n} = \frac{k_p^3 \bar{R}_l w}{2j\omega\varepsilon_0} S^l_{201,p\bar{n}n}(1+\delta_{p0}) \tag{A18}$$

$$Q^{(eh)}_{pl,\bar{n}n} = \frac{k_p^2 p}{2k_0^2} \left\{ S^l_{131,p\bar{n}n}(1-\delta_{p0}) - S^l_{302,p\bar{n}n}(1+\delta_{p0}) \right\} \tag{A19}$$

$$Q^{(hh)}_{pl,\bar{n}n} = \frac{k_p \bar{R}_l w}{2j\omega\mu_0} S^l_{210,p\bar{n}n}(1-\delta_{p0}) \tag{A20}$$

$$C^{(he)}_{pl,\bar{n}n} = S^{(he)}_{pl,\bar{n}n} = K^{(he)}_{pl,\bar{n}n} = Q^{(he)}_{pl,\bar{n}n} = 0. \tag{A21}$$

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Analysis of Metallic Waveguides of a Large Class of Cross Sections Using Polynomial Approximation and Superquadric Functions

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***Abstract*—By using the polynomial approximation and superquadric functions in the Rayleigh–Ritz procedure, a unified method has been proposed to analyze conducting hollow waveguides of a large class of cross sections in our previous paper. Some useful and complicated cross-sectional waveguides in the microwave system, namely, eccentric annular, pentagonal, L-shaped, single-ridged, and double-ridged waveguides are analyzed in this paper. Compared with other numerical methods, this method has the advantages of straightforward, accurate, and computational effective.**

***Index Terms*—Polynomial approximation, Rayleigh–Ritz method, superqaudric functions, waveguide analysis.**

I. INTRODUCTION

The analysis of a uniform metallic hollow waveguide can be carried out by solving the Helmholtz equation and matching boundary conditions on its cross section. A large number of techniques have been proposed in the literature for this purpose: one is the boundary integral–resonant mode expansion (BI-RME) [1]. By using superquadric functions [2], [3] to describe the boundary of the waveguide in the Rayleigh–Ritz method, various cross-sectional waveguides (including rectangular, circular, elliptic, coaxial, triangular, etc.) have been analyzed successfully in a unified manner [4]. In this paper, we extend the application of this method to analyze some waveguides with more complicated cross sections that are commonly used in microwave systems. The cross sections of various hollow metallic waveguides to be analyzed are shown in Fig. 1(a)–(f) for eccentric annular, pentagonal ($N = 4$ and $N = 5$), L-shaped, single-ridged, and double-ridged waveguides.

Analysis of eccentric annular waveguides has been a subject of numerous investigations [5], [6]. In [5], combined with conformal transformation, the method of intermediate problems was used to find the lower bounds and the Rayleigh–Ritz method to find the upper bounds of the cutoff frequency, both for TE and TM modes. A family of new waveguides, pentagonal waveguides [described by *ABCDE* in Figs. 1(b) and 1(c)], has been proposed in [7]. The conformal-mapping finite-difference (CMFD) method was used to analyze its propagation characteristics, and the computed data were compared to some measurement results. L-shaped, single-ridged, and double-ridged waveguides are formed from variations of the rectangular waveguide. They can be used in satellite communication systems for wide-bandwidth operations [8], [9]. The surface integral-equation method (SIE) [10], the finite-element method (FEM) [11]–[13], and the finite-difference method (FDM) [14], [15] have been used to study these structures.

The method in this paper does not need a complex mathematical manipulation (such as conformal mapping) and discretization procedure in the above methods. In Section II, a brief description of the algorithm is given. In Section III, numerical results obtained here are compared with those by other methods and measurement data. A conclusion is drawn in Section IV.

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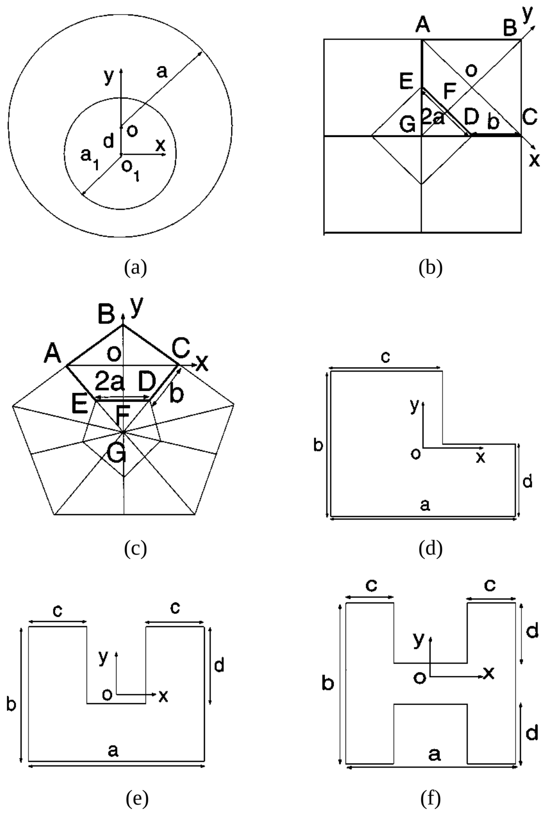

Fig. 1. Cross sections of: (a) eccentric annular, (b) pentagonal ($N = 4$), (c) pentagonal ($N = 5$), (d) L-shaped, (e) single-ridged, and (f) double-ridged waveguides.

II. Brief Description of the Algorithm

Details about this method can be found in [4] and [16]. The waveguide cross section is assumed to be in the x-y-plane, and z is the longitudinal direction in Cartesian coordinates. k_z denotes the wavenumber in the z-direction and the time variation $e^{j\omega t}$ is suppressed. The wave equation need to be solved is

$$\nabla_T^2 \begin{Bmatrix} E_z \\ H_z \end{Bmatrix} + k_c^2 \begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = 0 \qquad (1)$$

where ∇_T is the Laplacian operation in the x-y-plane. For TM and TE cases, the Dirichlet and Neumann boundary conditions need to be satisfied, respectively.

The longitudinal field E_z or H_z is approximated by a series of polynomials Φ_i

$$\begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = \sum_{i=1}^{m} c_i \Phi_i \qquad (2)$$

where m stands for the polynomial number. The generalized eigenvalue matrix equation is

$$\mathbf{KC^T} = \mathbf{k}_c^2 \mathbf{MC} \qquad (3)$$

where $\mathbf{C}$ is the column matrix of unknown coefficients, while matrices $\mathbf{K}$ and $\mathbf{M}$ are given in [16]. The cutoff wavenumber and modal spectrum of the waveguide can be obtained once (3) has been solved.

The more complicated the waveguide cross section, the more complex the electromagnetic fields in the waveguide, and more polynomials will be needed in the above method. Using the QZ factorization for the generalized eigenvalue problem [17], the eigenvalues and corresponding eigenvectors of (3) can be obtained accurately even with the large polynomial number m, which will result in very ill-conditioned matrices $\mathbf{K}$ and $\mathbf{M}$. For a specific problem, the maximum value of the polynomial number is reached when negative eigenvalue appears. The negative eigenvalue violates the system matrices' positive definite property.

TABLE I
Comparison of the Cutoff Wavenumbers for TM and TE Modes in an Eccentric Annular Waveguide ($a = 1, a_1 = 0.5$, and $d = 0.2$)

	TM case			TE case		
Mode	[5]	Present	ϵ	[5]	Present	ϵ
1	4.8119	4.8129	0.15	1.3522	1.3614	0.89
2	5.5125	5.5252	0.23	1.4097	1.4122	0.30
3	6.1735	6.2099	0.59	2.6840	2.7155	1.16
4	6.8002	6.8375	0.55	2.6862	2.7331	1.72
5	7.3957	7.4619	0.89	3.9298	3.9723	1.07
6	7.6919	8.0615	1.24	3.9298	3.9891	1.50
7	8.4991	8.6160	1.36	5.0192	5.0251	0.12
8	9.0106	9.1358	1.27	5.1138	5.1694	0.99
9	9.3488	9.3673	0.20	5.1139	5.1752	1.18
10	9.4763	9.6245	1.54	5.834	5.9336	1.68
11	9.9577	10.159	1.98	6.261	6.3051	0.70
12	10.107	10.236	1.26	6.262	6.3262	1.02
13	10.253	10.409	1.50	6.599	6.7580	2.74
14	10.848	11.113	2.38	7.311	7.4655	2.07
15	10.923	11.326	3.56	7.385	7.5345	1.98
16	11.557	11.460	0.86	7.392	7.6171	2.96
17	11.773	12.136	2.98	7.994	8.4071	4.91
18	12026	12.407	1.18	8.500	8.6461	1.69
19	12074	12.491	2.00	8.626	8.6984	0.79
20	12.93	13.017	0.67			

TABLE II
Comparison of the Cutoff Wavenumbers of TE Modes in an L-shaped Waveguide ($a = b = 1.27$ cm, and $c = d = 0.635$ cm)

				ϵ between Present and	
Mode	[14]	[10]	Present	[14]	[10]
TE_1	1.9111	1.8917	1.9653	2.76	3.74
TE_2	2.9600	2.9159	2.9632	0.11	1.60
TE_3	4.9452	4.8755	4.9475	0.05	1.46
TE_4	4.9452		4.9475	0.05	
TE_5	5.3128	5.2463	5.3198	0.13	1.38
TE_6	5.5799		5.6334	0.95	
TE_7	6.9937		6.9968	0.04	
TE_8	7.2784		7.7939	1.62	
TE_9	7.6002		7.6193	0.25	

III. Numerical Results

The cutoff wavenumbers or wavelengths are represented in corresponding Tables I–VI. The error (ϵ) in these tables means the relative error. For the TE case, due to the Neumann boundary condition, there exists a null mode with the zero cutoff wavenumber. It is a nonphysical mode and has been removed from the result.

A. Eccentric Annular Waveguides

The following function is used to describe the waveguide geometry

$$\psi(x,y) = \left(\left|\frac{x}{a}\right|^2 + \left|\frac{y}{a}\right|^2 - 1 \right) \left(\left|\frac{x}{a_1}\right|^2 + \left|\frac{y-d}{a_1}\right|^2 - 1 \right) \qquad (4)$$

where a is the radius of the outer circle, a_1 is the radius of the inner circle, and d is the eccentric distance.

The cutoff wavenumbers for both TM and TE modes in several eccentric annular waveguides have been computed and compared with the corresponding upper bounds in [5]. Reasonably agreements are observed for all the cases, but only one set of results is listed in Table I. The polynomial number used is 65 for both TM and TE cases.

TABLE III
COMPARISON OF THE CUTOFF WAVELENGTHS (IN CENTIMETERS) FOR PENTAGONAL WAVEGUIDE (EXAMPLE I: $N = 5$, $2a = 23$ mm, AND $b = 10.4$ mm)

				ε between Present and	
Mode	Measured [7]	CMFD [7]	Present	Measured [7]	CMFD [7]
TE_1	5.526	5.396	5.4207	1.94	2.41
TE_2	3.724	3.695	3.6784	1.25	0.78
TM_1	3.059	2.918	3.0917	1.06	4.83

TABLE IV
COMPARISON OF THE CUTOFF WAVELENGTHS (IN CENTIMETERS) FOR PENTAGONAL WAVEGUIDE (EXAMPLE II: $N = 4$, $2a = 72$ mm, AND $b = 34$ mm)

				ε between Present and	
Mode	Measured [7]	CMFD [7]	Present	Measured [7]	CMFD [7]
TE_1	18.601	18.297	18.4528	0.80	1.66
TE_2	14.497	14.434	14.1569	0.28	1.07
TM_1	10.709	10.526	10.5424	1.58	1.74

TABLE V
COMPARISON OF THE CUTOFF WAVENUMBERS OF TE MODES IN A SINGLE-RIDGED WAVEGUIDE ($a = 1.0$ cm, $b = 0.5$ cm, AND $c = d = 0.25$ cm)

	Results of Different Method			ε between Present and	
Mode	[14]	[10]	Present	[14]	[10]
TE_1	2.2422	2.2496	2.2772	1.54	1.21
TE_2	4.8543	4.9436	4.9705	2.34	0.54
TE_3	6.4476	6.5189	6.5120	0.99	0.01
TE_4	7.5185	7.5642	7.5252	0.09	0.52
TE_5	9.8311		9.8792	0.42	
TE_6	12.5607		12.5664	0.05	
TE_7	12.5665		12.5607	0.05	
TE_8	12.7820		12.7667	0.12	
TE_9	13.4243		13.3823	0.31	

TABLE VI
COMPARISON OF THE CUTOFF WAVENUMBERS OF TE MODES FOR DOUBLE-RIDGED WAVEGUIDE ($a = 1.27$ cm, $b = 1.016$ cm $c = 0.508$ cm, AND $d = 0.3683$ cm)

	Results of Different Methods					ε between Present and			
Mode	[11]	[12]	[13]	[14]	Present	[11]	[12]	[13]	[14]
$TE_{10}H$	1.440	1.439	1.437	1.434	1.4849	3.02	3.09	3.23	3.43
$TE_{10}T$				3.168	3.2015				1.09
$TE_{20}T$	6.192	6.193	6.197	6.192	6.2065	0.23	0.22	0.15	0.23
$TE_{30}H$	6.713	6.714	6.721	6.705	6.7230	0.15	0.13	0.03	0.27
$TE_{11}T$				6.975	7.0021				0.39

B. Pentagonal Waveguides

For pentagonal waveguides, the constraint function that is used to describe the waveguide boundary can be written as

$$\psi(x,y) = \left(\left|\frac{x}{a}\right| + \frac{y}{b_1} - 1\right)\left(\left|\frac{x}{a}\right| + \frac{y}{b_2} - 1\right)(y + b_3) \qquad (5)$$

where $a = OC$, $b_1 = OB$, $b_2 = OF$, and $b_3 = OG$ in Figs. 1(b) and (c). Cutoff wavelengths of two samples waveguides were measured and computed by CMFD [7]. The waveguide parameters of example I [as shown in Fig. 1(b)] are $N = 5$, $2a = 23$ mm, and $b = 10.4$ mm; the waveguide example II [as shown in Fig. 1(c)] has parameters $N = 4$, $2a = 72$ mm and $b = 34$ mm. The comparison among the present results and the measurement data and the CMFD is shown in Tables III and IV. For the TE case, the polynomial number is 40, and it is 25 for the TM case in the computation.

C. L-Shaped , Single-Ridged, and Double-Ridged Waveguides

For L-shaped, single-ridged, and double-ridged waveguides, only TE modes are considered in this paper. The matrices in (3) can be evaluated by the same numerical integration route used for all cases with different input parameters. The result of the L-shaped waveguide with dimensions $a = b = 1.27$ cm and $c = d = 0.635$ cm are shown in Table II. The polynomial number used is 75. The analytic solution for some modes exist for above waveguide [14]. In the finite-difference–simultaneous iteration with Chebyshev acceleration (FD–SIC) [14] method, it has found that the TE_3 mode has double degeneracy. In the present method, it also shows this phenomenon.

The result of a single-ridged waveguide with parameters $a = 1.0$ cm, $b = 0.5$ cm, and $c = d = 0.25$ cm is shown in Table V. The polynomial number is 75 for this structure. The results of the SIE [10] and FD–SIC [14] methods are also included in this table for comparison.

As the last example, the symmetric double-ridged waveguide with dimensions $a = 1.27$ cm, $b = 1.016$ cm, $c = 0.508$ cm, and $d = 0.3683$ cm, depicted in Fig. 1(f), is analyzed. The polynomial number is 95 for the double-ridged waveguide. The result of the present method for y-symmetric (due to the even function of y) TE modes, and the comparison with the results from [11]–[14] are also given in Table VI. In all of the above tables, satisfactory agreements are observed.

IV. CONCLUSIONS

The cutoff wavenumbers or wavelengths of the TE and TM modes for the eccentric annular, pentagonal, L-shaped, single-ridged, and double-ridged waveguides have been reinvestigated in a unified manner in this paper by using the unified method proposed in [4]. Through the numerical results, the accuracy, efficiency, and flexibility of the present method are demonstrated. Similar to the FD–SIC method [14], the present method can also find some missing modes in literature. Based on the works in [4] and this paper, we can conclude that a large class of waveguides can be analyzed in a unified manner by this method efficiently.

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